Int. J. Heat Mass Transfer. Vol. 28, No. 9, pp. 1783–1786, 1985
Printed in Great Britain

0017-9310/85 \$3.00+0.00

TECHNICAL NOTES

Mixed convection in porous media adjacent to a vertical uniform heat flux surface

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(*Received* 22 *May* 1984 *and in final form* 25 *February* 1985)

NOMENCLATURE

$d(x)$	surface temperature variation resulting from pure natural convection boundary-layer solution, $Nx^{1/3}$
f	nondimensional stream function
F_2	nondimensional correction in stream function due to higher-order boundary-layer effects
g	acceleration due to gravity
K	permeability of the pourous medium
k	thermal conductivity of the saturated porous medium
N	$[q''^2\mu\alpha/k^2\rho_\infty g\beta K\{-\phi_0'(0)\}^2]^{1/3}$
Nu_x	local Nusselt number with mixed convection and higher-order boundary-layer corrections
$(Nu_x)_{b\text{-}l}$	local Nusselt number resulting from the leading-order natural convection boundary-layer solution
Pe_x	local Peclet number, $U_\infty x/\alpha$
q''	constant heat flux at the wall
r	$(x^2+y^2)^{1/2}$
$\bar{R}$	constant defined in equation (8)
Ra_x	local Rayleigh number, $\rho g\beta x\,Kd(x)/\mu\alpha$
t	temperature
u	velocity component in x direction
U_c	characteristic natural convection velocity, $(\alpha/x)\,Ra_x$
U_∞	free-stream velocity
x	vertical coordinate
y	horizontal coordinate.

Greek symbols

α	thermal diffusivity of porous medium
β	coefficient of thermal expansion of fluid
ρ	density of fluid
ψ	stream function
θ	angular coordinate measured from $+x$ axis
ε_M	perturbation parameter due to mixed convection
ε_H	perturbation parameter due to higher-order boundary-layer correction
μ	dynamic viscosity of fluid
ϕ	nondimensional temperature, $(t-t_\infty)/(t_o-t_\infty)_0$
Φ_2	nondimensional correction in temperature due to higher-order boundary-layer effects
λ	$\left[\dfrac{\rho g\beta KN}{\mu\alpha}\right]^{-3/4}$
η	nondimensional horizontal coordinate, $y/x(Ra_x)^{1/2}$.

Subscripts

o	condition at $y=0$
0	condition when ε_H and ε_M are zero
1	first-order correction due to mixed convection
2	second-order correction due to mixed convection
∞	value in the ambient
r	reference value.

INTRODUCTION

THE MECHANISMS of mixed convection in porous media have important applications in the utilization of geothermal energy. The injection or removal of geothermal fluids causes pressure gradients. These pressure gradients in turn give rise to an imposed external flow.

Similarity solutions for some vertical and inclined mixed convection boundary region flows in porous media have been obtained by Cheng [1]. Previous work on mixed convection in porous media is also summarized there. Self-similar solutions are obtained only when both the surface temperature and the forced convection velocity are of the power-law form downstream and their exponents are identical. An isothermal vertical surface satisfies these conditions for a uniform free stream. However, for matching exponents, the uniform flux surface must be inclined at an angle of 45° from the horizontal, for a uniform free stream.

Merkin [2] has considered mixed convection boundary-layer flow in porous media adjacent to a vertical uniform heat flux surface. Two different coordinate perturbations were developed for small and large downstream distances. These expansions were then matched for the intermediate downstream distances, using numerical integration of the governing equations.

The following analysis is applicable for mixed convection flow from a vertical uniform heat flux surface, at large downstream distances from the leading edge. Using the method of matched asymptotic expansions, it is shown that the first correction to the boundary-layer theory, neglected by Merkin [2], occurs at the same level as the second correction due to mixed convection. Both these effects, therefore, must be included in a physically consistent analysis. A similar situation was encountered by Carey and Gebhart [3] for mixed convection in Newtonian media. The present analysis also gives details of changes in the outer irrotational flow, resulting from modification of the inner flow due to mixed convection effects.

ANALYSIS

We consider the aiding mixed convection flow past a vertical uniform flux surface in a porous medium, where x and y are the downstream and normal coordinates, and r and θ are the plane radial and angular coordinates, respectively. The governing equations with the relevant boundary conditions are:

continuity equation:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{1}$$

Darcy's law:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{K}{\mu}\rho_r g\beta \frac{\partial}{\partial y}(t - t_\infty) \tag{2}$$

energy equation:

$$\frac{\partial \psi}{\partial y}\frac{\partial t}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial t}{\partial y} = \alpha\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) \tag{3}$$

$$q'' = -k\frac{\partial t}{\partial y}, \quad v = 0 \quad \text{at} \quad y = 0 \quad \text{at} \quad x > 0$$

$$t \sim t_\infty, \quad u \sim u_\infty \quad \text{as} \quad r \to \infty, \quad \theta \neq 0 \qquad \text{(4a, b, c)}$$

$$\frac{\partial u}{\partial y} = 0 = v = \frac{\partial t}{\partial y} \quad \text{at} \quad y = 0, \quad x < 0.$$

Far downstream from the leading edge, the situation corresponds to a natural convection flow, weakly perturbed by an external flow. The appropriate parameter governing the mixed convection effects is found to be

$$\varepsilon_M = \frac{U_\infty}{U_c} = \frac{Pe_x}{Ra_x}. \tag{5}$$

In equation (5) U_c is the characteristic velocity of the natural convection flow and Pe_x is the local Peclet number. Moreover, the higher-order boundary-layer corrections are governed by

$$\varepsilon_H = \frac{1}{\sqrt{Ra_x}} \tag{6}$$

ε_M and ε_H are then seen to be related as:

$$\varepsilon_M = \bar{R}\sqrt{\varepsilon_H} \tag{7}$$

where $\bar{R}$ is a constant given by

$$\bar{R} = Pe_x Ra_x^{-3/4} = \left[\frac{\mu\{-\phi_o'(0)\}k}{\alpha q''\rho_\infty g\beta K}\right]^{1/2} U_\infty. \tag{8}$$

In the following analysis $\bar{R}$ has been assumed to be of O(1). An *a priori* decision about the order of $\bar{R}$ is required for the ordering of terms in the subsequent expansions. For the inner region, we next postulate the following two parameter expansions up to $O(\varepsilon_H)$

$$\psi = \alpha Ra_x^{1/2}[f_0(\eta) + \bar{R}\varepsilon_H^{1/2} f_1(\eta) + \bar{R}^2\varepsilon_H f_2(\eta) + \varepsilon_H F_2(\eta) + \cdots] \tag{9}$$

and

$$(t - t_\infty) = d(x)[\phi_0(\eta) + \bar{R}\varepsilon_H^{1/2}\phi_1(\eta) + \bar{R}^2\varepsilon_H\phi_2(\eta) + \varepsilon_H\Phi_2(\eta) + \cdots]. \tag{10}$$

Alternatively, (9) and (10) can be written in terms of ε_M.

The zeroth-order terms in (9) and (10) correspond to the pure natural convection boundary-layer flow obtained in [4]. The additional terms are effects due to mixed convection, f_1, f_2 and ϕ_1, ϕ_2 and corrections to the boundary-layer theory, F_2 and Φ_2. It is seen from (9) and (10) that the second-order mixed convection effects, f_2 and ϕ_2, arise at the same level as the first corrections to the boundary-layer theory, F_2 and Φ_2.

In the outer inviscid and irrotational region

$$\psi = \tilde{\psi}_o + \tilde{\psi}_1 + \tilde{\psi}_2 + \cdots \tag{11}$$

$$t = t_\infty. \tag{12}$$

Substituting (9) and (10) into (2) and (3) yields for f_1 and ϕ_1

$$f_1'' - \phi_1' = 0 \tag{13}$$

$$\phi_1'' - \tfrac{1}{3}f_1'\phi_o + \tfrac{1}{3}f_1\phi_o' + \tfrac{2}{3}f_o\phi_1 = 0. \tag{14}$$

The boundary condition for f_1' as $\eta \to \infty$ is determined by considering the outer solution, which is to the zeroth level

$$\tilde{\psi}_o = U_\infty y = U_\infty r \sin\theta. \tag{15}$$

Matching with the inner solution yields

$$f_1'(\infty) = 1. \tag{16}$$

The complete boundary conditions for (13) and (14) then are

$$f_1(0) = \phi_1'(0) = 1 - f_1'(\infty) = \phi_1(\infty) = 0. \tag{17}$$

From (13) and (17), f_1' and ϕ_1 are seen to be related as

$$f_1' = 1 + \phi_1. \tag{18}$$

The inner expansions at the next level give

$$f_2'' - \phi_2' = 0 \tag{19}$$

$$\phi_2'' + \tfrac{1}{3}f_o'\phi_2 - \tfrac{1}{3}f_2'\phi_o + \tfrac{1}{3}f_1\phi_1' + \tfrac{2}{3}f_o\phi_2' = 0. \tag{20}$$

The outer expansion is again considered, to obtain the boundary conditions as $\eta \to \infty$.

$$\nabla^2\tilde{\psi}_1 = 0; \quad \tilde{\psi}_1|_{\theta=\pi} = 0, \quad \tilde{\psi}_1|_{\theta=0} = \alpha f_o(\infty)\left(\frac{r}{\lambda}\right)^{2/3} \tag{21}$$

which results in

$$\tilde{\psi}_1 = \frac{\sqrt{3}}{2}\left(\frac{r}{\lambda}\right)^{2/3} f_o(\infty)\sin[\tfrac{2}{3}(\pi - \theta)]. \tag{22}$$

Matching this with the inner solution yields:

$$f_2'(\infty) = 0, \quad F_2'(\infty) = -\frac{2}{3\sqrt{3}}\eta f_o(\infty). \tag{23}$$

The complete set of boundary conditions for (18) and (19) then becomes

$$f_2(0) = \phi_2'(0) = f_2'(\infty) = \phi_2(\infty) = 0. \tag{24}$$

From (19) and (24)

$$f_2' = \phi_2. \tag{25}$$

The first boundary-layer corrections F_2, ϕ_2 have been determined in Joshi and Gebhart [5].

The second-order outer solution is then governed by

$$\nabla^2\tilde{\psi}_2 = 0; \quad \tilde{\psi}_2|_{\theta=\pi} = 0, \quad \tilde{\psi}_2|_{\theta=0} = [f_2(\infty)\bar{R}^2 + A] \tag{26}$$

where A is a constant given by

$$A \equiv \lim_{\eta\to\infty}\left[F_2(\eta) + \frac{2}{3\sqrt{3}}\eta f_o(\eta)\right]. \tag{27}$$

To the expansions (9) and (10), eigenfunctions can still be added which satisfy the governing equations (2) and (3) and homogeneous boundary conditions. The first eigenvalue for the expansions (9) and (10) occurs at $\varepsilon_H^{3/2}$, which is identical to the pure natural convection problem treated in Joshi and Gebhart [5]. Thus (9) and (10) are valid up to $O(\varepsilon_H)$. It is worth noting that the third correction due to mixed convection occurs at $O(\varepsilon_H^{3/2})$ in (9) and (10) which is at the same level as the first eigenfunction. Logarithmic terms must then be added to the expansions (9) and (10) at $O(\varepsilon_H^{3/2})$ for correct behavior as $\eta \to \infty$. This situation has been encountered by Merkin [6] in a study of mixed convection from isothermal vertical plates in Newtonian media.

RESULTS

Equations (13), (14) and 19), (20) with the proper boundary conditions were solved numerically using

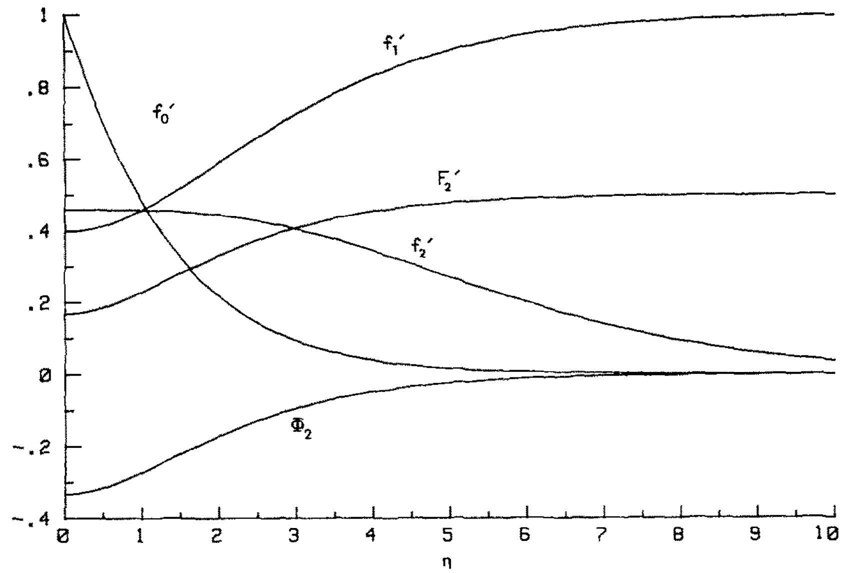

FIG. 1. Variation of zero-order velocity function f_o' and higher-order corrections f_1', f_2', F_2' and Φ_2 with nondimensional horizontal coordinate η.

Hamming's predictor–corrector method. The correction functions, f_0', f_1', f_2', F_2' and Φ_2, as functions of η are plotted in Fig. 1. ϕ_1 and ϕ_2 can then be determined from (18) and (25).

The surface temperature decay up to $O(\varepsilon_H)$ is given by

$$(t_o - t_\infty) = (t_o - t_\infty)_0[1 - 0.603\bar{R}\varepsilon_H^{1/2} + 0.459\bar{R}^2\varepsilon_H - 0.333\varepsilon_H] \quad (28)$$

where $(t_o - t_\infty)_0$ is the decay resulting from the pure natural convection boundary-layer solution and is given by

$$(t_o - t_\infty)_0 = [q''^2\mu\alpha/(k^2\{\phi_o'(0)\}^2\rho g\beta K)]^{1/3}x^{1/3}.$$

The $O(\varepsilon_H^{1/2})$ mixed convection correction in equation (28) is identical to that in Merkin [2], after correcting for an algebraic error in his expression for surface temperature. The $O(\varepsilon_H)$ correction in Merkin [2] does not include the last term in (28).

The local Nusselt number accurate up to $O(\varepsilon_H)$ is given by

$$Nu_x \equiv \frac{hx}{k} = \frac{q''x}{k(t_o - t_\infty)_0[1 - 0.603\bar{R}\varepsilon_H^{1/2} + 0.459\bar{R}^2\varepsilon_H - 0.333\varepsilon_H]}. \quad (29)$$

Using (29), Nu_x can be compared with $(Nu_x)_{b\text{-}l}$, the pure natural convection convection Nusselt number resulting from

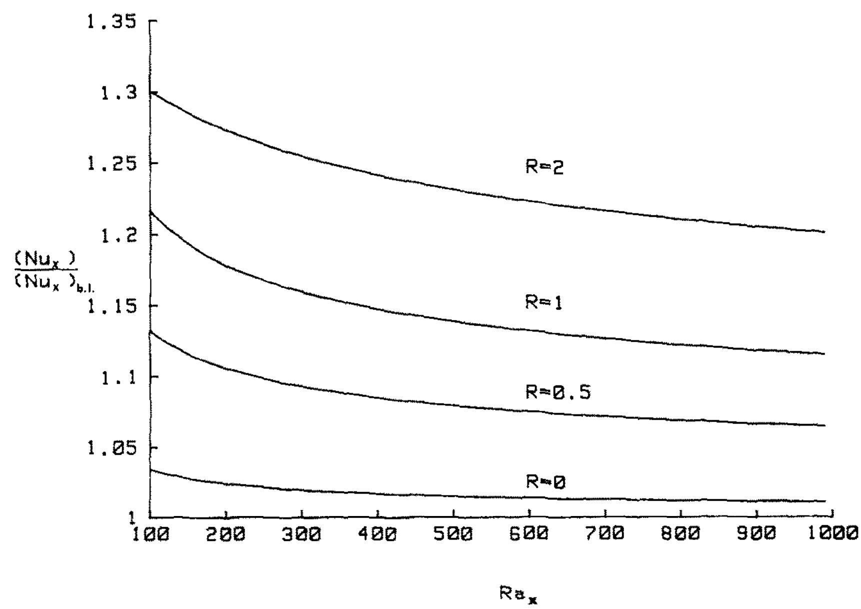

FIG. 2. Ratio of the local Nusselt number to its zero-order natural convection boundary-layer estimate, as a function of Ra_x, for various R.

the leading-order boundary-layer solution:

$$\frac{Nu_x}{(Nu_x)_{b\text{-}l}} = \frac{1}{[1-0.603\bar{R}\varepsilon_H^{1/2}+0.459\bar{R}^2\varepsilon_H-0.333\varepsilon_H]}. \quad (30)$$

Figure 2 shows the variation of $Nu_x/(Nu_x)_{b\text{-}l}$ with $Ra_x = 1/(\varepsilon_H)^2$, for various $\bar{R}$, using (30). The $\bar{R} = 0$ curve applies in the absence of an external free stream, when the higher-order natural convection boundary-layer corrections up to $O(\varepsilon_H)$ are taken into account. Variations for $\bar{R} = 0.5$, 1 and 2 are for progressively increasing mixed convection effects. From Fig. 2 it is seen that an increase in $\bar{R}$ results in a higher value of the ratio $Nu_x/(Nu_x)_{b\text{-}l}$. It is clear that even moderate values of the mixed convection parameter can result in a 10–25% increase in the local Nusselt number in the range $100 \leqslant Ra_x < 1000$.

Acknowledgements—This study was carried out under National Science Foundation grant MEA8200613. The expert efforts of Ms Anita Jackson in preparation of the manuscript are appreciated.

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Int. J. Heat Mass Transfer. Vol. 28, No. 9, pp. 1786–1787, 1985
Printed in Great Britain
0017–9310/85 $3.00 + 0.00
Pergamon Press Ltd.

Ambiguities related to the calculation of radiant heat exchange between a pair of surfaces

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(*Received* 23 *March* 1985)

NOMENCLATURE

- A surface area
- E_b black-body emissive power
- $\mathscr{F}$ Hottel's script-F factor
- F angle factor
- $Q_{1,2}$ radiant heat exchange between surfaces 1 and 2 in Oppenheim's definition, equation (1)
- Q_{1-2} radiant heat exchange between surfaces 1 and 2 in Hottel's definition, equation (4)
- T absolute temperature.

Greek symbol
- ε emissivity.

INTRODUCTION

IN THE ANALYSIS of radiant transfer, the desired end result is the net heat transfer at a given surface due to interchange with all surfaces with which it can interact radiatively. Frequently, as an intermediate step in the analysis, the value of the radiant exchange between a pair of surfaces is determined, and the values are summed for all pairs which involve the surface of interest, thereby yielding its net heat transfer. In various heat transfer textbooks, there appears to be some unwitting confusion about the radiant exchange between surface pairs. The objective of this note is to illuminate those issues.

First, it may be noted that the two most commonly used procedures for the analysis of radiant interchange, the Oppenheim network method [1] and Hottel's script-F method [2], yield different results for the radiant exchange between surface pairs. This fact appears not to have been realized before. For example, in [3–5], the pair-specific radiant exchange as given by the two procedures is improperly indicated as being the same. This may be verified by comparing equations (11.56) and (11.60) in [3], equations (11.2) and (11.23) in [4], and equations (11.57) and (11.67) in [5].

Second, in [6], the formula for the net heat transfer between the surfaces in a two-surface enclosure is mistakenly employed to evaluate the radiant exchange between a pair of surfaces which do not form an enclosure.

PAIR-SPECIFIC RADIANT EXCHANGE

In terms of the radiosity, Oppenheim [1] defined the radiant heat exchange between two surfaces 1 and 2 as

$$Q_{1,2} = \frac{J_1 - J_2}{1/(A_1 F_{12})} \quad (1)$$

where A is the radiative surface area and F is the shape factor. For an enclosure of N surfaces, the radiosities at all of the surfaces $i = 1, 2, \ldots, N$ can be determined by solving the following set of N algebraic equations

$$J_i = \varepsilon_i E_{bi} + (1-\varepsilon_i) \sum_{j=1}^{N} F_{ij} J_j, \quad i = 1, 2, \ldots, M \quad (2)$$

$$J_i = \sum_{j=1}^{N} F_{ij} J_j, \quad i = (M+1), \ldots, N \quad (3)$$

where the temperatures are prescribed at surfaces $i = 1$ through M, while surfaces $(M+1)$ through N are no-flux surfaces.

On the other hand, Hottel [2] defined the radiant heat exchange between surfaces 1 and 2 as

$$Q_{1-2} = A_1 \mathscr{F}_{12}(E_{b1} - E_{b2}) = -A_2 \mathscr{F}_{21}(E_{b2} - E_{b1}) \quad (4)$$

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